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OPEN CHANNELS WITH NONUNIFORM DISCHARGE

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PAPERS

OPEN CHANNELS WITH NONUNIFORM DISCHARGE

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Synopsis

Steady flows in an open channel with the addition of water along the course of flow are encountered frequently in engineering practice. Examples of this condition are the wash-water troughs in filters, the effluent channels around sewage-treatment tanks, and side-channel spillways. An equation based on the principle of conservation of momentum is derived in this paper. For cases in which the loss of energy because of friction is of secondary importance, solutions are obtained for prismatic channels of various cross sections and slopes. The hydraulic behavior of the flow is described with nondimensional quantities, and the validity of the theory is verified by tests on model channels.

Introduction

Notation.—The letter symbols adopted for use in this paper are defined where they first appear, in the illustrations or in the text, and are arranged alphabetically, for convenience of reference, in the Appendix.

The hydraulic behavior of an open channel with the addition of water along the course of flow is quite different from that of a channel with uniform discharge. During uniform discharge, energy is lost entirely because of frictional and form resistance. During nonuniform discharge an appreciable part of the loss of energy results from the impact between the added water and the water flowing in the channel. In many cases, the energy loss because of impact is of primary importance.

The Momentum Equation

Because the loss of energy as a result of impact in channels with nonuniform discharge is not known, an equation based on the conservation of energy is not useful in this case. The principle of conservation of momentum, however, is applicable. The momentum equation has been used successfully in the study of hydraulic jumps in straight channels having a moderate slope.

In order to obtain a relatively simple momentum equation, the following must be assumed:

1. The flow is considered to be uni-directional, but actually there are strong cross currents in the case of a side-channel spillway. However, in investigating the flow along the channel, the lateral unevenness of the water surface, as a result of cross currents, can be neglected.

2. The velocity in the direction of flow is uniformly distributed across each cross section.

3. The pressure in the flow is considered to be hydrostatic. This assumption is valid except for a short distance near the outlet end of the channel where the curvature of the water surface can be appreciable.

4. The average shearing stress at the channel walls is considered to be identical, at the same depth, as the stress for the same discharge in uniform flow.

A volume of water bounded by two sections a distance dx apart is shown in Fig. 1. The rate of momentum, in the direction of the flow, supplied to this volume is the sum of $\frac{\rho\, Q^2}{A}$ and $\rho\, q\, v'\, dx$. The rate of momentum emitted from this volume is $\frac{\rho\, (Q + dQ)^2}{A + dA}$. The difference between these two rates of momentum is equal to the force acting on the volume in the direction of the

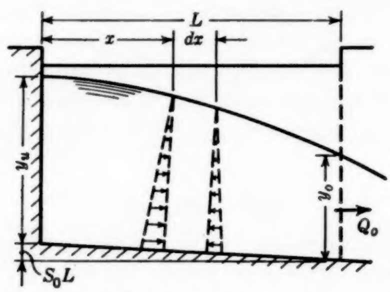

FIG. 1.—OPEN CHANNEL WITH NONUNIFORM DISCHARGE

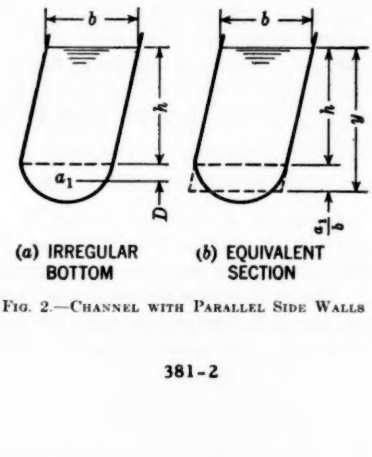

FIG. 2.—CHANNEL WITH PARALLEL SIDE WALLS

flow. Thus, neglecting the small effect of the slope on the pressure, and that of the entrained air on the volume of flow,

$$\rho\, g\, \bar{y}\, A - \rho\, g\, (\bar{y} + d\bar{y})\, (A + dA) + \rho\, g\, A\, S_0\, dx - \frac{\rho\, g\, Q^2\, P}{A^2\, C^2}\, dx = \frac{\rho\, (Q + dQ)^2}{A + dA} - \frac{\rho\, Q^2}{A} - \rho\, q\, v'\, dx \quad (1)$$

in which ρ is the mass per unit volume of the water, g denotes gravitational acceleration, $\bar{y}$ is the vertical distance from the centroid of the water cross section to the water surface, A represents the area of the cross section, S_0 is the slope of the channel, Q denotes the rate of discharge, P is the wetted perimeter, C is the Chezy coefficient, q represents the additional discharge per unit length of the channel, and v' is the component of the velocity of the added water in the direction of the channel flow.

For cases in which the frictional loss is of secondary importance, the shearing force at the channel walls can be considered to be balanced by the momentum of the added water. Then,

$$\rho\, g\, \bar{y}\, A - \rho\, g\, (\bar{y} + d\bar{y})\, (A + dA) + \rho\, g\, A\, S_0\, dx = \frac{\rho\, (Q + dQ)^2}{A + dA} - \frac{\rho\, Q^2}{A} \quad (2)$$

When water is added uniformly along the channel,

$$Q = \frac{Q_o\, x}{L} \quad (3a)$$

and

$$(Q + dQ) = \frac{Q_o\, (x + dx)}{L} \quad (3b)$$

With the Froude number $\mathbf{F}_o$ defined as

$$\mathbf{F}^2_o = \frac{Q^2_o\, b_o}{g\, A^3_o} \quad (3c)$$

Eq. 2 can be reduced to

$$-\, \bar{y}\, A^2\, dA - A^3\, d\bar{y} + A^3\, S_0\, dx = \frac{2\, \mathbf{F}^2_o\, A^3_o\, A\, x\, dx}{b_o\, L^2} - \frac{\mathbf{F}^2_o\, A^3_o\, x^2\, dA}{b_o\, L^2} \quad (3d)$$

in which b is the width of the water surface in the channel. The subscripts o, u, and c are used to signify values in the channel at the outlet end, the upstream end, and the section with critical flow, respectively.

Eqs. 1, 2, and 3*d* are differential equations from which a closed-form solution cannot be obtained in the general case. However, for a particular channel flow, the profile of the water surface in the channel can be obtained by numerical integration. Eq. 3*d* will be used subsequently for investigating channels with parallel side walls and channels with side walls set at a constant slope (triangular and trapezoidal sections). The results presented will enable engineers to design channels without the tedious process of numerical integration.

CHANNELS WITH PARALLEL SIDE WALLS

Channels are to be considered as having uniform slope, along which water is to be added uniformly, and in which the energy loss resulting from friction is negligible. Fig. 2(a) represents a channel having parallel sides and a bottom of arbitrary shape. The centroid of the area a_1 is located as shown, and the water surface is considered to lie above this area.

In Fig. 2(a),

$$A = b\,h + a_1 \quad \text{(4a)}$$

and

$$A\,\bar{y} = \tfrac{1}{2}\,b\,h^2 + a_1\,(h + D) \quad \text{(4b)}$$

from which

$$\bar{y} = \frac{A}{2\,b} + \frac{1}{A}\left(a_1\,D - \frac{a^2_1}{2\,b}\right) \quad \text{(5a)}$$

$$d\bar{y} = \left[\frac{1}{2\,b} - \frac{1}{A^2}\left(a_1\,D - \frac{a^2_1}{2\,b}\right)\right] dA \quad \text{(5b)}$$

Substituting Eqs. 4 and 5 into Eq. 3d and simplifying,

$$\left[\mathsf{F}^2_o\left(\frac{x}{L}\right)^2 - \left(\frac{A}{A_o}\right)^3\right] d\left(\frac{A}{A_o}\right) = \left[2\,\mathsf{F}^2_o\left(\frac{A}{A_o}\right)\left(\frac{x}{L}\right) - \left(\frac{A}{A_o}\right)^3 \frac{S_0\,L}{\left(\frac{A_o}{b}\right)}\right] d\left(\frac{x}{L}\right) \quad \text{(6)}$$

Reducing the channel section to a flat-bottomed section of equal area, and letting y denote the depth of flow measured from the resulting flat bottom (Fig. 2(b)),

$$\frac{A}{A_o} = \frac{b\,y}{b\,y_o} = \frac{y}{y_o} \quad \text{(7a)}$$

$$\left[\mathsf{F}^2_o\left(\frac{x}{L}\right)^2 - \left(\frac{y}{y_o}\right)^3\right] d\left(\frac{y}{y_o}\right) = \left[2\,\mathsf{F}^2_o\left(\frac{y}{y_o}\right)\left(\frac{x}{L}\right) - G\left(\frac{y}{y_o}\right)^3\right] d\left(\frac{x}{L}\right) \quad \text{(7b)}$$

in which G is defined as $S_0\,L/\left(\frac{A_o}{b_o}\right)$ and in this case is equal to $S_0\,L/y_o$.

Channels with Level Bottom.—For a level-bottomed channel, S_0 equals zero and Eq. 7b reduces to

$$\frac{d\left(\frac{x}{L}\right)^2}{d\left(\frac{y}{y_o}\right)} - \frac{\left(\frac{x}{L}\right)^2}{\frac{y}{y_o}} = -\frac{\left(\frac{y}{y_o}\right)^2}{\mathsf{F}^2_o} \quad \text{(8)}$$

With the boundary conditions y/y_o and x/L both equal to unity at the outlet end of the channel, the solution of Eq. 8 is

$$2\,\mathsf{F}^2_o\left(\frac{x}{L}\right)^2 = (2\,\mathsf{F}^2_o + 1)\,\frac{y}{y_o} - \left(\frac{y}{y_o}\right)^3 \quad \text{(9a)}$$

At the upstream end of the channel (where $x = 0$),

$$\frac{y_u}{y_o} = \sqrt{2\,\mathsf{F}^2_o + 1} \qquad (9b)$$

By the use of Eq. 9*a* the entire profile of the water surface in the channel can be determined from the flow condition at the outlet end. When free fall occurs at the outlet end, the flow is critical, and therefore $\mathsf{F}_o = 1$. The depth y_o can be computed through the knowledge of Q_o. When the outlet end is submerged y_o is fixed by the downstream surface elevation, and F_o can be computed by knowing y_o and Q_o. There are two real positive solutions for y/y_o from Eq. 9*a* for each value of x/L. Because the energy of a water particle must decrease as it proceeds downstream, only the values of $y/y_o \geqq 1$ are true solutions for the problem of level channels.

To demonstrate the validity of the momentum equation, a 9-in.-wide rectangular channel with a flat bottom was used. The length of the channel was varied from 7.50 ft to 4.38 ft. Water was added to the channel over the level tops of the two side walls. The outlet end of the channel was connected to a similar channel 6 ft long, and no water was added to this part of the flow. The value of F_o at the outlet end of the upper channel was computed by observing Q_o and y_o, and was found to be 0.64 for all rates of discharge. The observed surface profiles in the upper channel at different rates of discharge are shown in Table 1 with the values computed from Eq. 9*a* shown in paren-

TABLE 1.—DEPTH OF FLOW IN LEVEL RECTANGULAR CHANNELS[a]

Channel length, in feet	Discharge in cubic feet per second	Value of $\frac{X}{L}$				
		0.033	0.266	0.444	0.633	0.772
7.50	3.06	1.40(1.46)	1.39(1.44)	1.35(1.41)	1.31(1.35)	1.18(1.28)
	2.52	1.25(1.28)	1.25(1.27)	1.20(1.24)	1.14(1.18)	1.05(1.12)
	2.26	1.19(1.19)	1.18(1.18)	1.14(1.15)	1.07(1.10)	1.02(1.05)
	2.14	1.11(1.15)	1.11(1.13)	1.07(1.11)	1.03(1.06)	0.94(1.01)
	1.85	1.05(1.04)	1.04(1.03)	1.01(1.01)	0.95(0.96)	0.89(0.91)
	1.60	0.98(0.95)	0.98(0.94)	0.95(0.91)	0.91(0.88)	0.85(0.83)
	1.25	0.82(0.81)	0.79(0.80)	0.79(0.78)	0.75(0.74)	0.70(0.71)
		Value of $\frac{X}{L}$				
		0.022	0.258	0.510	0.701	
5.63	2.15	1.10(1.15)	1.11(1.14)	1.07(1.10)	1.01(1.04)	
	1.64	0.98(0.96)	0.98(0.95)	0.99(0.92)	0.94(0.87)	
	1.39	0.88(0.86)	0.87(0.85)	0.84(0.82)	0.80(0.78)	
	0.88	0.62(0.63)	0.61(0.63)	0.62(0.60)	0.58(0.57)	
		Value of $\frac{X}{L}$				
		0.048	0.372	0.623		
4.38	2.50	1.25(1.27)	1.24(1.24)	1.18(1.18)		
	2.47	1.27(1.27)	1.22(1.23)	1.11(1.17)		
	1.90	1.11(1.06)	1.02(1.04)	1.01(0.99)		
	1.87	1.05(1.05)	1.02(1.03)	0.99(0.98)		
	1.52	0.88(0.91)	0.89(0.89)	0.82(0.85)		
	1.50	0.90(0.91)	0.90(0.89)	0.85(0.84)		
	1.06	0.69(0.72)	0.71(0.70)	0.65(0.67)		

[a] Values shown in parentheses represent depths computed from Eq. 9*a*. Other values represent experimentally observed depths in feet.

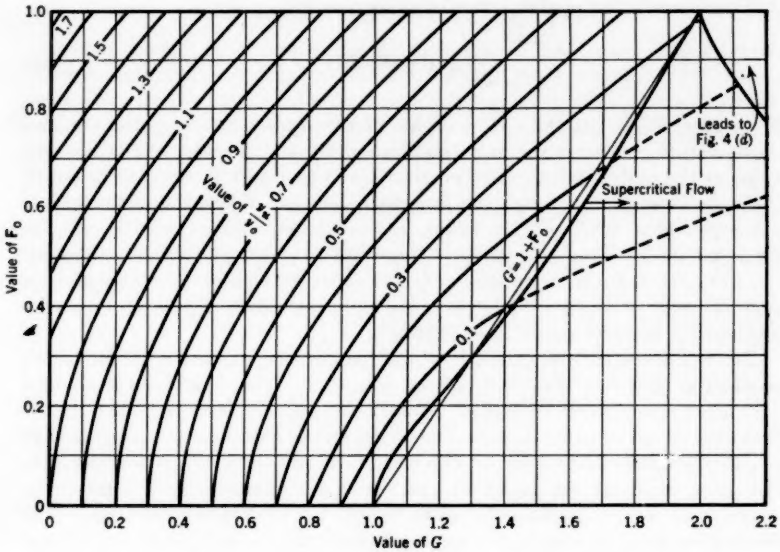

FIG. 3.—SOLUTIONS FOR CHANNELS WITH PARALLEL SIDE WALLS

theses. Thus, the corresponding observed and computed values are in agreement. The application of the momentum equation to this case is therefore acceptable.

Channels with Bottom at a Uniform Slope.—For channels with a sloping bottom, a closed-form solution of Eq. 7*b* cannot be obtained. Except for cases with a hydraulic jump in the channel, the surface profile can be obtained from a known depth of flow by numerical integration. For cases with $\mathbf{F}_o \leqq 1$, values of y_u/y_o have been obtained from the known condition that y/y_o equals unity at x/L equal to one, and are shown as solid lines in Fig. 3. The values of y_u/y_o are shown because the water surface slopes downward in a downstream direction and only the value of y_u is of interest in determining the dimensions of the channel.

The different conditions of channel flow are shown in Fig. 4. From Eq. 7*b* it can be seen that, if G is less than $\frac{2}{3}$ $(1 + 2\,\mathbf{F}^2_o)$, then $d\mathbf{F}/dx$ exceeds zero—that is, the value of $\mathbf{F}$ increases as the flow proceeds downstream as shown in Fig. 4(*a*). The dashed lines in Figs. 4(*a*), 4(*b*), 4(*c*), and 4(*d*) indicate the depths at which $\mathbf{F}$ is equal to unity.

When G is greater than $\frac{2}{3}$ $(1 + 2\,\mathbf{F}^2_o)$ but less than approximately $(1 + \mathbf{F}_o)$, the value of $\mathbf{F}$ first increases as the flow proceeds downstream to reach a maximum value and then $\mathbf{F}$ decreases, as shown in Fig. 4(*b*). The line dividing regions B and C in Fig. 4(*e*) (corresponding to Fig. 4(*b*) and 4(*c*), respectively) includes all the cases in which the maximum value of $\mathbf{F}$ in the channel is equal to unity. This line can be represented approximately by the equation, $G = 1 + \mathbf{F}_o$.

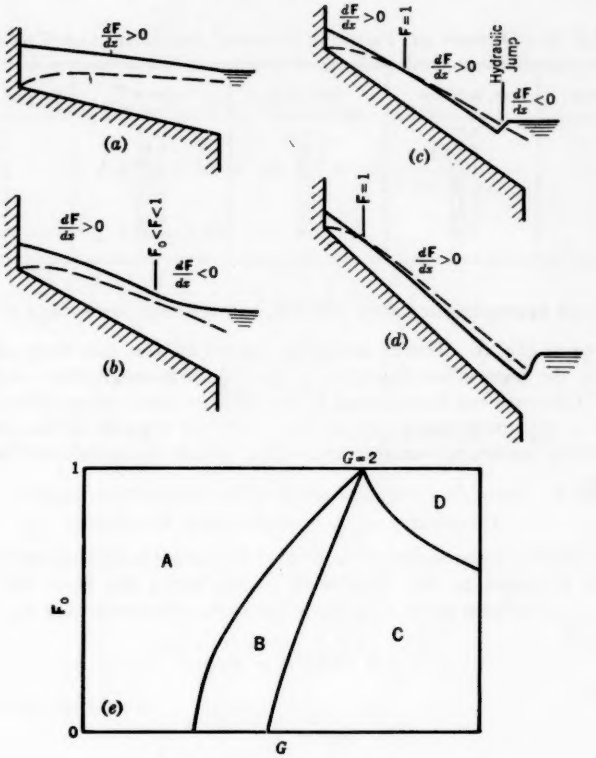

FIG. 4.—TYPES OF FLOW AS DETERMINED BY $\mathbf{F}_o$ AND G

When G is greater than approximately $(1 + \mathbf{F}_o)$, there is supercritical flow in a part of the channel. If the outlet is sufficiently submerged, a hydraulic jump will form as shown in Fig. 4(*c*). In this case, the control point is shifted into the channel, and the elevation of the water surface at the outlet does not affect the entire profile in the channel. Thus, the surface profile above the hydraulic jump cannot be determined from the value of y_o.

Fig. 4(*d*) shows the case in which the depth of submergence is not great enough to create an hydraulic jump in the channel. The flow is supercritical throughout the lower part of the channel. In this case, the value of $\mathbf{F}_o$ is not determined by the depth of submergence at the outlet.

If the outlet is not submerged (Fig. 4(*a*)), $\mathbf{F}_o$ will be equal to unity if G is less than or equal to 2. When G is greater than 2, the flow is supercritical at the lower part of the channel. Cases involving supercritical flow will be investigated subsequently.

To verify the theory for sloping channels, a model rectangular channel, 3 in. wide and 4 ft 6 in. long, with S_0 equal to 13% was used. Water was added uniformly along the flow over a weir on one side of the channel. The outlet end was free so that $\mathbf{F}_o$ was equal to unity when G was less than 2. The depths y_u and y_o were measured at different rates of discharge. The results obtained are shown in Table 2. The observed values of y_u/y_o and

TABLE 2.—DEPTHS OF FLOW IN SLOPING RECTANGULAR CHANNEL

y_o, in inches	y_u, in inches	Value of G	Value of $\frac{y_u}{y_o}$	Value of $\frac{S_0 L}{y_o} + \frac{y_u}{y_o}$
5.00	2.75	1.40	0.55	1.95
4.88	2.00	1.44	0.41	1.85
4.50	1.50	1.56	0.33	1.89
4.25	1.25	1.65	0.30	1.95
4.00	1.25	1.75	0.31	2.06
3.75	1.12	1.87	0.30	2.17
3.62	1.00	1.93	0.28	2.21
3.50	1.00	2.00	0.29	2.29

$\frac{S_0 L}{y_o} + \frac{y_u}{y_o}$ are approximately 20% and 5%, respectively, lower than the values obtained from Fig. 3. This is probably caused by the fact that, at such a steep slope, the term involving v' in Eq. 1 is no longer negligible. At a lesser slope, the difference can be expected to be much smaller. Since it is the value of $S_0 L + y_u$ that determines the required depth of channel at the outlet, the verification is considered satisfactory. The results presented will lead to a safe design.

CHANNELS WITH SLOPING SIDE WALLS

Fig. 5 shows a cross section of a prismatic channel with sloping side walls. The water is added to the channel uniformly along the flow, the channel bottom is at a uniform slope, and the water surface is above area a_2.

In Fig. 5

$$A = \tfrac{1}{2} k h^2 - a_2 \qquad (10a)$$

from which

$$h = \sqrt{\frac{2 (A + a_2)}{k}} \qquad (10b)$$

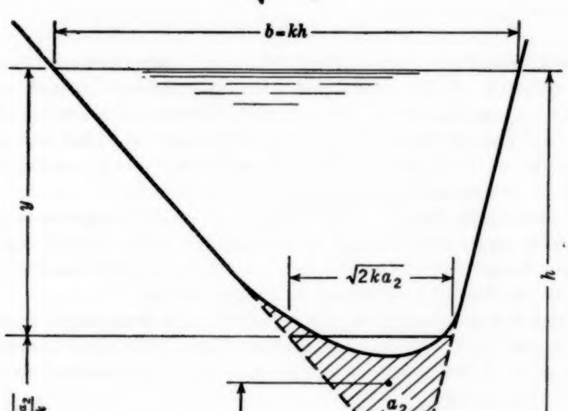

FIG. 5.—CHANNEL WITH SLOPING SIDE WALLS

Also,

$$A\,\bar{y} = \tfrac{1}{6}\,k\,h^3 - a_2\,(h - D) \qquad (10c)$$

in which D locates the centroid of the area a_2. Thus,

$$A\,d\bar{y} + \bar{y}\,dA = d(A\,\bar{y}) = \frac{A\,dA}{\sqrt{2\,k\,(A + a_2)}} \qquad (10d)$$

Substituting Eq. 10*d* into Eq. 3*d*,

$$-\sqrt{\frac{1 + B^2}{\frac{A}{A_o} + B^2}}\left(\frac{A}{A_o}\right)^3 d\left(\frac{A}{A_o}\right) + G\left(\frac{A}{A_o}\right)^3 d\left(\frac{x}{L}\right)$$

$$= 2\,\mathsf{F}^2_o\left(\frac{A}{A_o}\right)\left(\frac{x}{L}\right) d\left(\frac{x}{L}\right) - \mathsf{F}^2_0\left(\frac{x}{L}\right)^2 d\left(\frac{A}{A_o}\right) \qquad (11)$$

in which B denotes $\sqrt{\frac{a_2}{A_o}}$, and G is equal to $\frac{S_0\,L\,\sqrt{2\,k\,(A_o + a_2)}}{A_o}$ in this case.

Level Triangular Channels.—For level triangular channels, B and G are equal to zero. Since

$$\frac{A}{A_o} = \left(\frac{y}{A_o}\right)^2 \qquad (12a)$$

and

$$d\left(\frac{A}{A_o}\right) = 2\left(\frac{y}{y_o}\right) d\left(\frac{y}{y_o}\right) \qquad (12b)$$

Eq. 11 can be reduced to

$$\frac{d\left(\frac{x}{L}\right)^2}{d\left(\frac{y}{y_o}\right)} - \frac{2}{\frac{y}{y_o}}\left(\frac{x}{L}\right)^2 = -\frac{2}{\mathsf{F}^2_o}\left(\frac{y}{y_o}\right)^4 \qquad (13a)$$

With the boundary condition y/y_o equals one at x/L equal to one, the solution of Eq. 13*a* is

$$3\,\mathsf{F}^2_o\left(\frac{x}{L}\right)^2 = (3\,\mathsf{F}^2_o + 2)\left(\frac{y}{y_o}\right)^2 - 2\left(\frac{y}{y_o}\right)^5 \qquad (13b)$$

There are two positive real solutions for y/y_o for each value of x/L. Because of energy considerations, only the values of y/y_o greater than unity are the true solutions. At the upstream end of the channel, where $x = 0$,

$$\frac{y_u}{y_o} = \sqrt[3]{\tfrac{3}{2}\,\mathsf{F}^2_o + 1} \qquad (14)$$

Sloping Triangular Channels.—For triangular channels with a uniform bottom slope, the various types of flow occurring in the channel are similar to those shown in Fig. 4. In this case, it can be seen from Eq. 11 that the equation for the line between regions A and B is

$$G = \tfrac{4}{5}\,(1 + \tfrac{3}{2}\,\mathsf{F}^2_o) \qquad (15a)$$

and the line between regions B and C is

$$G = 2 \qquad \text{(15b)}$$

The case with supercritical flow in the channel ($G > 2$) will be investigated subsequently.

A closed-form solution for Eq. 11 with B^2 equal to zero cannot be obtained. For cases with G less than 2, the method of numerical integration is used and the resulting values of y_u/y_o for different values of F_o and G are presented as solid lines in Fig. 6. It should be noted that, in this case, $G = \frac{2\,S_0\,L}{y_o}$.

Level Channels with Sloping Side Walls.—With G equal to zero, Eq. 11 can be reduced to

$$\frac{d\left(\frac{x}{L}\right)^2}{d\left(\frac{A}{A_o}\right)} - \frac{\left(\frac{x}{L}\right)^2}{\frac{A}{A_o}} = -\frac{\sqrt{1 + B^2}}{\mathsf{F}^2_o}\,\frac{\left(\frac{A}{A_o}\right)^2}{\sqrt{\frac{A}{A_o} + B^2}} \qquad (16)$$

With the boundary condition that A/A_o equals one at x/L equal to one, the solution of Eq. 16 is

$$\left(\frac{x}{L}\right)^2 = \frac{A}{A_o}\left\{1 + \frac{2\sqrt{1 + B^2}}{3\,\mathsf{F}^2_o} \times \left[(1 + B^2)^{3/2} - \left(\frac{A}{A_o} + B^2\right)^{3/2}\right] - \frac{2\,B\sqrt{1 + B^2}}{\mathsf{F}^2_o} \times \left[(1 + B^2)^{1/2} - \left(\frac{A}{A_o} + B^2\right)^{1/2}\right]\right\} \qquad (17)$$

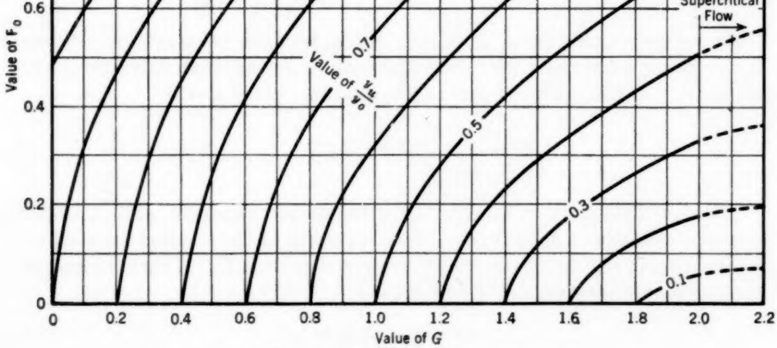

FIG. 6.—SOLUTIONS FOR TRIANGULAR CHANNELS

Eq. 17 yields the surface profile in the channel.

At the upstream end of the channel (where x equals zero) the value of A_u/A_o can be obtained from

$$1 + \frac{2\sqrt{1 + B^2}}{3\,\mathbf{F}^2_o}\left[(1 + B^2)^{3/2} - \left(\frac{A_u}{A_o} + B^2\right)^{3/2}\right] - \frac{2B\sqrt{1 + B^2}}{\mathbf{F}^2_o}\left[(1 + B^2)^{1/2} - \left(\frac{A_u}{A_o} + B^2\right)^{1/2}\right] = 0 \quad \text{(18)}$$

If the depth y is measured from the bottom of an equivalent trapezoidal section, as shown in Fig. 5, y_u/y_o can be determined from

$$\frac{y_u}{y_o} = \frac{\sqrt{\dfrac{A_u}{A_o} + B^2} - B}{\sqrt{1 + B^2} - B} \quad \text{(19)}$$

This process of obtaining a solution for y_u/y_o is tedious. A suggested empirical formula giving results to within 1% of those obtained by use of Eq. 19 is

$$\frac{y_u}{y_o} = \left(\frac{y_u}{y_o}\right)_r - \left(\frac{1}{3}\right)^B\left[\left(\frac{y_u}{y_o}\right)_r - \left(\frac{y_u}{y_o}\right)_t\right] \quad \text{(20)}$$

in which $\left(\frac{y_u}{y_o}\right)_r$ is the value of $\frac{y_u}{y_o}$ for a rectangular section having the same $\mathbf{F}_o$ as given by Eq. 9*b*, and $\left(\frac{y_u}{y_o}\right)_t$ is the value of $\frac{y_u}{y_o}$ for a triangular section by Eq. 14.

Sloping Channels with Sloping Side Walls.—The various types of flow that will occur in a sloping channel with sloping side walls are similar to those shown in Fig. 4. In this case, the line between regions B and C is represented approximately by the expression:

$$G = 2 - 2\,(1 - \mathbf{F}_o)\left(\sqrt{B^2\,(1 + B^2)} - B^2\right) \quad \text{(21)}$$

For actual values of G greater than those determined from Eq. 21 there will be supercritical flow in the channel.

For cases not involving supercritical flow, the profile of the water surface in the channel for given values of $\mathbf{F}_o$, G, and B^2 can be obtained from Eq. 11 by numerical integration. A close approximation of y_u/y_o can be obtained by first finding y_u/y_o for a rectangular channel with the same values of $\mathbf{F}_o$ and G from Fig. 3, and the same for a triangular channel from Fig. 6. The value of y_u/y_o for the channel in question can be obtained from the empirical formula:

$$\frac{y_u}{y_o} = \left(\frac{y_u}{y_o}\right)_r - \left(\frac{1}{3}\right)^B\left[\left(\frac{y_u}{y_o}\right)_r - \left(\frac{y_u}{y_o}\right)_t\right] \quad \text{(22)}$$

in which $\left(\frac{y_u}{y_o}\right)_r$ and $\left(\frac{y_u}{y_o}\right)_t$ denote, respectively, the values of $\frac{y_u}{y_o}$ for a rectangular channel and for a triangular channel, with the same values of $\mathbf{F}_o$ and G as the given channel.

Channels with Supercritical Flow

It has been stated that when the actual value of G exceeds the value determined by Eq. 21, there will be supercritical flow in the channel. Also, if the outlet end is sufficiently submerged, an hydraulic jump will form. It should be noted that Eq. 21 will change to $G > 1 + \mathsf{F}_o$ and $G > 2$ for channels with parallel walls and for triangular channels, respectively. When free fall occurs at the outlet, F_o is equal to one if G is equal to or less than 2, and F_o exceeds unity if G exceeds 2 for all channels.

Flow above Critical Section.—If x_c locates the section with critical flow, and since

$$\frac{Q_c}{Q_o} = \frac{x_c}{L} \qquad (23a)$$

and

$$\mathsf{F}^2{}_c = \frac{Q^2{}_c\, b_c}{g\, A^3{}_c} = 1 \qquad (23b)$$

then

$$\frac{Q^2{}_o\, b_c}{g\, A^3{}_c}\left(\frac{x_c}{L}\right)^2 = 1 \qquad (23c)$$

If the upper part of the channel is considered as a separate channel of length x_c, $G_c = \dfrac{S_0\, x_c}{\frac{A_c}{b_c}}$ by definition. Since F_c is equal to unity, G_c cannot be greater than 2. However, G_c cannot be smaller than 2 because this would make $\dfrac{S_0\, x}{\frac{A}{b}}$ less than 2 at the sections immediately downstream, and the flow there would be subcritical. Therefore,

$$\frac{S_0\, x_c}{\frac{A_c}{b_c}} = 2 \qquad (24a)$$

or

$$\frac{x_c}{L} = \frac{2\, A_c}{S_0\, L\, b_c} \qquad (24b)$$

Substituting Eqs. 24 into Eq. 23*c*,

$$A_c\, b_c = \left(\frac{2\, Q_o}{\sqrt{g}\, S_0\, L}\right)^2 \qquad (25)$$

From Eq. 25 y_c can be determined. The value of y_c being known, x_c can be computed from Eq. 24*b*. According to Figs. 3 and 6 and Eq. 22, the value of y_u is given by

$$\frac{y_u}{y_c} = 0.31 - \left(\frac{1}{3}\right)^E (0.31 - 0.63) \qquad (26)$$

in which E is equal to $\sqrt{a_2/A_c}$.

Flow below Critical Section—Without Hydraulic Jump.—For a steep channel without a hydraulic jump (Fig. 4(*d*)) the outlet condition is independent of the depth of submergence at the outlet. The surface profile below the section with critical flow can be obtained by numerical integration.

For this purpose, Eq. 3d can be reduced to

$$\left[\left(\frac{x}{x_c}\right)^2 - \sqrt{\frac{1+E^2}{\frac{A}{A_c}+E^2}}\left(\frac{A}{A_c}\right)^3\right] d\left(\frac{A}{A_c}\right)$$

$$= 2\left[\frac{x}{x_c}\frac{A}{A_c} - \left(\frac{A}{A_c}\right)^3\right] d\left(\frac{x}{x_c}\right) \quad \text{..(27)}$$

The boundary condition is A/A_c equals one at x/x_c equal to one. The values of A/A_c at various (x/x_c)-values have been computed and are shown in Fig. 7. It should be noted that the values of A/A_c are practically the same for various values of E^2. To determine the outlet condition for a given channel flow, x_c/L is computed from Eq. 24b and the value for A_o/A_c is equal to the value of A/A_c at $x/x_c = L/x_c$. The values of y_o and F_o can then be computed from the value of A_o.

Flow below Critical Section with Hydraulic Jump.—For a steep channel with an hydraulic jump (Fig. 4(*c*)) the profile of the water surface in the supercritical sections can be determined from Fig. 7. At the outlet the minimum depth of submergence required to produce an hydraulic jump is equal to the conjugate depth for the values of F_o and y_o applicable to the case without the jump. The values of F_o and G computed from this minimum depth of submergence form the line dividing region C and the region leading to the situation shown in Fig. 4(*d*). A depth of submergence greater than this minimum depth will move the hydraulic jump upstream.

When there is an hydraulic jump in the channel, the values of F_o, G, and y_o are determined by the depth of submergence at the outlet. A convenient

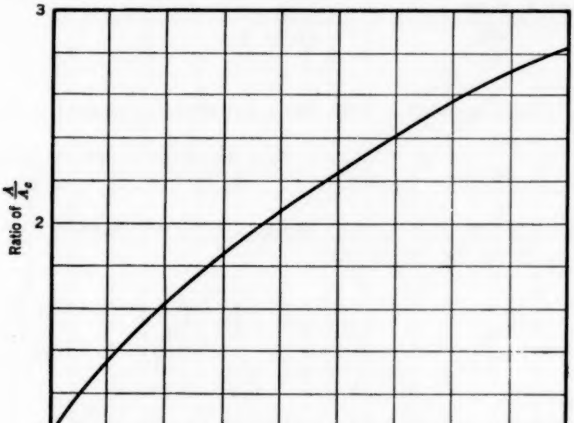

Fig. 7.—Surface Profile in Supercritical Sections

expression for finding y_u can be obtained by combining Eqs. 25 and 26. Thus, for channels with parallel side walls,

$$\frac{y_u}{y_o} = 1.24 \left(\frac{\mathsf{F}_o}{G}\right)^2 \text{.....................} (28a)$$

and for triangular channels,

$$\frac{y_u}{y_o} = \left(\frac{\mathsf{F}_o}{G}\right)^{\frac{4}{5}} \text{.......................} (28b)$$

Eqs. 28 have been plotted as dashed lines in Figs. 3 and 6.

Efect of Friction

The momentum of the added water and the friction on the channel walls have been neglected in the foregoing examples. For hydraulic structures the momentum balances the friction in sloping channels, and the results presented will lead to a safe design. However, when the channel is level or at a very gradual slope, it is necessary to investigate the effect of friction on the depth of channel flow. The result of this investigation will serve to indicate the order of magnitude of the effect of friction for channels set at a very gradual slope.

When the channel bottom is level and water is added uniformly along the channel, Eq. 1 is reduced to

$$-\,\bar{y}\,A^2\,dA - A^3\,d\bar{y} - \frac{Q^2_o\,P\,x^2\,dx}{C^2\,L^2} = \frac{2\,\mathsf{F}^2_o\,A^3_o\,A\,x\,dx}{b_o\,L^2} - \frac{\mathsf{F}^2_o\,A^3_o\,x^2\,dA}{b_o\,L^2} \text{..} (29)$$

If h_f denotes the frictional loss of head that would result if the total channel discharge, Q_o, were flowing at the outlet depth y_o for a distance equal to the channel length L, then

$$h_f = \frac{Q^2_o\,P_o\,L}{C^2_o\,A^3_o} \text{.......................} (30)$$

according to Chezy formula. With the approximation that $C = C_o$, Eq. 30 can be rewritten as

$$-\,\bar{y}\,A^2\,dA - A^3\,d\bar{y} - \frac{h_f\,A^3_o\,P\,x^2\,dx}{P_o\,L^3} = \frac{2\,\mathsf{F}^2_o\,A^3_o\,A\,x\,dx}{b_o\,L^2} - \frac{\mathsf{F}^2_o\,A^3_o\,x^2\,dA}{b_o\,L^2} \text{..} (31)$$

Level Channels with Parallel Side Walls.—For level channels with parallel side walls, Eq. 31 can be reduced to

$$\left[1 + \tfrac{1}{2}\,\frac{h_f}{\mathsf{F}^2_o\,y_o}\,\frac{P}{P_o\sqrt{\dfrac{y}{y_o}}}\,\frac{\dfrac{x}{L}}{\sqrt{\dfrac{y}{y_o}}}\right]\frac{d\left(\dfrac{x}{L}\right)^2}{d\left(\dfrac{y}{y_o}\right)} - \frac{\left(\dfrac{x}{L}\right)^2}{\dfrac{y}{y_o}} = -\,\frac{\left(\dfrac{y}{y_o}\right)^2}{\mathsf{F}^2_o} \text{.....} (32)$$

This nonlinear differential equation should be solved by successive approximations. The first approximation is obtained for frictionless flow by neglecting the term involving h_f; this results in Eq. 9a. From Eq. 9a, the following can be obtained

$$\frac{\dfrac{x}{L}}{\sqrt{\dfrac{y}{y_o}}} = \sqrt{1 - \frac{1}{2\,\mathsf{F}^2_o}\left[\left(\frac{y}{y_o}\right)^2 - 1\right]} \text{...............} (33)$$

The value of $\left(\frac{y}{y_o}\right)^2$ changes along the channel, is equal to unity at the outlet, and is equal to $(2\,\mathbf{F}^2_o + 1)$ at the upstream end, according to Eq. 9*b*. Thus the value of $\frac{x}{L} \Big/ \sqrt{\frac{y}{y_o}}$ varies from zero at the upstream end to unity at the outlet. To find y_u (for practical purposes), a constant value of $\frac{1}{2}$ can be used for $\frac{x}{L} \Big/ \sqrt{\frac{y}{y_o}}$ throughout the channel. It can also be shown that the value of $\frac{P}{P_o}\sqrt{\frac{y}{y_o}}$ remains very close to unity throughout a channel of reasonable proportion. Thus, Eq. 32 can be linearized for finding y_u:

$$\left[1 + \frac{1}{4}\,\frac{h_f}{\mathbf{F}^2_o\, y_o}\right]\frac{d\left(\frac{x}{L}\right)^2}{d\left(\frac{y}{y_o}\right)} - \frac{\left(\frac{x}{L}\right)^2}{\frac{y}{y_o}} = -\,\frac{\left(\frac{y}{y_o}\right)^2}{\mathbf{F}^2_o} \qquad (34)$$

The solution of Eq. 34 gives the following value of y_u at the upstream end:

$$\frac{y_u}{y_o} = \left(1 + \frac{8 + 3\,\alpha}{4}\,\mathbf{F}^2_o\right)^I \qquad (35)$$

in which α denotes $\frac{h_f}{\mathbf{F}^2_o\, y_o}$ and I equals $\frac{4 + \alpha}{8 + 3\,\alpha}$. With h_f equal to zero for frictionless flow, Eq. 35 is identical to Eq. 9*b*.

The percentage increase of y_u because of friction, obtained by comparing Eqs. 9*b* and 35, is shown in Fig. 8. It can be verified that for wash-water troughs and side-channel spillways, the increase of y_u as a result of friction is

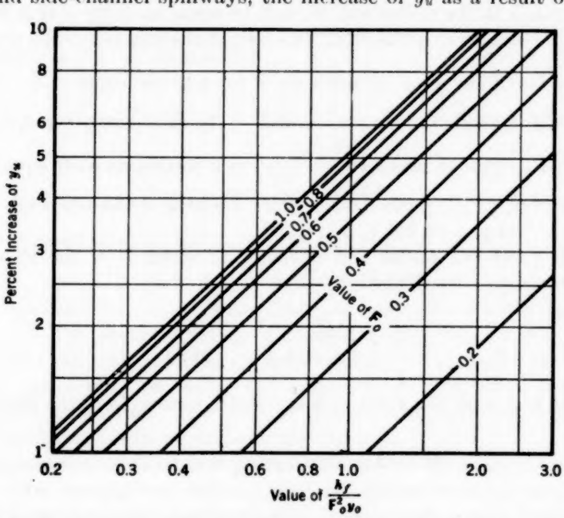

FIG. 8.—INCREASE OF y_u AS A RESULT OF FRICTION IN LEVEL CHANNELS

usually negligible. For effluent channels around sewage-treatment tanks, the value of $\frac{h_f}{\mathbf{F}^2_o\, y_o}$ can be as high as 2, and the increase of y_u caused by friction can become as large as 10%.

Level Channels with Sloping Side Walls.—With the same approximation as was applied to the case of channels with parallel side walls, Eq. 31 yields the following result for level triangular channels:

$$\frac{y_u}{y_o} = \left(1 + \frac{6 + 5\,\alpha}{4}\,\mathbf{F}^2_o\right)^J \qquad (36)$$

in which $J = \frac{2 + \alpha}{6 + 5\,\alpha}$. With h_f equal to zero for frictionless flow, Eq. 36 reduces to Eq. 14.

The percentage increase of y_u caused by friction has been computed from Eqs. 14 and 36 and is found to be accurately represented by Fig. 8. Thus, Fig. 8 is applicable to triangular channels and to channels with parallel side walls. Furthermore, since these two channel shapes are the two extreme cases of channels with sloping side walls, Fig. 8 can be applied safely to all channel shapes.

Examples of Application

The application of the preceding results will be illustrated by several examples.

Example 1.—If it is required to design a rectangular wash-water trough 20 ft long to discharge 5 cu ft per sec with a free fall at the outlet end, the channel requiring the least amount of material for construction must be found.

With a free fall at the outlet end, $\mathbf{F}_o$ will be equal to unity if G is less than or equal to 2. For a given value of y_o, the area of the two side walls is equal to $\left(\frac{2y_u}{y_o} + G\right) y_o\, L$. Since y_u/y_o depends on G for a given value of $\mathbf{F}_o$, the area of the side walls depends on the choice of G. By examining Fig. 3 it can be found that for $\mathbf{F}_o$ equal to unity $\left(2\,\frac{y_u}{y_o} + G\right)$ is a minimum at G equal to 1.7. For this value of G, y_u/y_o equals 0.45. Thus, the area of the side walls is equal to $(2 \times 0.45 + 1.7)\, y_o\, L$ or $2.6\, y_o\, L$.

The depth y_o at the outlet end depends on width b of the channel, $\mathbf{F}_o$ being equal to unity. By definition,

$$y_o = \left(\frac{Q_o}{\sqrt{g}\,\mathbf{F}_o\, b}\right)^{\frac{2}{3}} = \frac{0.92}{b^{\frac{2}{3}}} \qquad (37)$$

Neglecting the end wall, the total wall and bottom area A_t of the channel is

$$A_t = 2.6 \times 0.92 \times \frac{L}{b^{\frac{2}{3}}} + b\, L \qquad (38)$$

If the bottom and side walls are of the same thickness, the amount of material required for construction will be proportional to A_t. To find the condition for

the least value of A_t let dA_t/db equal zero.

$$\frac{dA_t}{db} = -2.6 \times 0.92 \times L \times \tfrac{2}{3}\, b^{-5/3} + L = 0 \qquad (39a)$$

$$b = 1.32 \text{ ft} \qquad (39b)$$

$$y_o = \frac{0.92}{b^{\frac{2}{3}}} = 0.76 \text{ ft} \qquad (39c)$$

$$S_0 = \frac{G\, y_o}{L} = \frac{1.7 \times 0.76}{20} = 0.065 = 6.5\% \qquad (39d)$$

$$y_u = 0.45\, y_o = 0.34 \text{ ft} \qquad (39e)$$

$$\text{Depth of channel at outlet end} = y_u + S_0 L = 1.64 \text{ ft} \qquad (39f)$$

$$A_t = 2.6\, y_o L + b L = 66.2 \text{ sq ft} \qquad (39g)$$

Example 2.—If it is required to design a level rectangular trough with a free fall at the outlet (for the same conditions as in Example 1), then $\mathbf{F}_o$ equals unity, and from Eq. 9*b*,

$$y_u = \sqrt{2\,\mathbf{F}^2_o + 1} \qquad y_o = 1.73\, y_o \qquad (40a)$$

By the use of Eq. 37,

$$y_u = 1.73 \times \frac{0.92}{b^{\frac{2}{3}}} \qquad (40b)$$

The total area A_t of the side walls and bottom is

$$A_t = 2\, y_u L + b L = 2 \times 1.73 \times \frac{0.92}{b^{\frac{2}{3}}} \times L + b L = \frac{3.18}{b^{\frac{2}{3}}} + b L \qquad (40c)$$

For a minimum A_t, $\frac{dA_t}{db}$ equals zero:

$$-3.18 \times \tfrac{2}{3} \times L \times b^{-5/3} + L = 0 \qquad (41a)$$

$$b = 1.57 \text{ ft} \qquad (41b)$$

$$y_o = \frac{0.92}{b^{\frac{2}{3}}} = 0.68 \text{ ft} \qquad (41c)$$

$$\text{Depth of channel} = y_u = 1.73\, y_o = 1.18 \text{ ft} \qquad (41d)$$

$$A_t = 2\, y_u L + b L = 78.6 \text{ sq ft} \qquad (41e)$$

On comparing the results with those in Example 1, these facts become apparent: (*a*) The required depth of channel at the outlet end is less with a level channel than with a sloping channel. When the available head is limited, a level channel should be used. (*b*) A sloping channel requires slightly less material for construction. The cost of construction, however, may be lower with a level channel because of its simpler layout.

It is apparent that there is nothing to be gained by using sloping channels for wash-water troughs and effluent channels around sewage-treatment tanks. A level channel consumes less hydraulic head, and the cost is comparable to that of a sloping channel.

Example 3.—This example will show the method of determining the performance of a trapezoidal channel of a side channel spillway 400 ft long, discharging 16,000 cu ft per sec. The bottom slope of the channel is 15%, the bottom width is 10 ft, side slopes are 2:1 (the area a_2 being equal to 50 sq ft). The outlet end of the channel is connected to a channel of similar cross section at a supercritical slope.

Since the downstream channel is at supercritical slope, the outlet end of the side channel is not submerged. The value of $\mathbf{F}_o$ can be greater than or equal to unity depending on whether the value G is greater or less than 2. If $\mathbf{F}_o$ equals unity, then (from Eq. 3c) y_o equals 28 ft, making $G = 3.4$. Therefore, $\mathbf{F}_o$ is greater than unity, y_o is less than 28 ft, G exceeds 3.4, and there will be supercritical flow in the channel.

From Eq. 25, y_c equals 17.3 ft when A_c is equal to 323 sq ft and b_c equals 27.3 ft. From Eq. 24b the critical depth occurs at x_c equal to 158 ft. From Eq. 26, y_u equals 10.0 ft. At the outlet L/x_c is equal to 2.53. From Fig. 7, A_o/A_c equals 1.50, or A_o is equal to 485 sq ft, resulting in y_o equal to 22.7 ft. These results can be compared with those obtained for a similar channel.[2]

[2] "Side Channel Spillways: Hydraulic Theory, Economic Factors, and Experimental Determination of Losses," by Julian Hinds, *Transactions*, ASCE, Vol. 89, 1926, p. 881.

Example 4.—This example illustrates the method of designing the channel of a side-channel spillway 400 ft long discharging 16,000 cu ft per sec. The maximum allowable side slope is 2:1, and the outlet end of the side channel is connected to a channel of similar cross section at a supercritical slope. The cost of construction is affected by the amount of earthwork and lining. For channels on sloping hillsides, it is usually more economical to use steep side slopes and small bottom widths.[2]

Let it be decided to use the maximum allowable side slope (2:1). A bottom width of 10 ft is chosen for trial. This choice makes the area a_2 equal 50 sq ft. The choice of bottom slope is guided by the empirical expression:

$$\sqrt{2\,\mathbf{F}^2_o + 1} < G < 2 \qquad (42)$$

Since the outlet end is not submerged, and G is chosen to be less than 2, for example, 1.8, the value of $\mathbf{F}_o$ is equal to unity. From Eq. 3c, y_o equals 28 ft when A_o is equal to 672 sq ft and b_o equals 38 ft. Thus, $S_0 = \dfrac{G\,A_o}{L\,b_o} = 8\%$.

To find y_u, Eq. 22 is used. For $\mathbf{F}_o$ equal to unity and G equal to 1.8, $\left(\dfrac{y_u}{y_o}\right)_r$ equals 0.40 from Fig. 3, and $\left(\dfrac{y_u}{y_o}\right)_t$ equals 0.67 from Fig. 6. With B^2 equal to 0.0745, Eq. 22 results in y_u/y_o equal to 0.60 and y_u equal to 16.8 ft. With y_u and S_0 determined, the cost of construction can be estimated.

Several bottom widths and G-values should be tried in order to obtain the most economical channel. It can be shown that the channel determined in this example with its gradual bottom slope involves only about two thirds as much earthwork as the channel described in Example 3.

Summary

A differential equation based on the principle of the conservation of momentum has been derived for channels with nonuniform discharge. Straight prismatic channels with water added uniformly along the flow have been investigated.

The effect of friction on the channel flow has been investigated, and the result is shown in Fig. 8. In many cases the effect of friction can be neglected, and the hydraulic behavior of the flow depends only on three dimensionless quantities which describe the outlet condition, the cross-sectional shape, and the slope of the channel. The different types of channel flow are shown in Fig. 4.

The solutions for channels with parallel side walls are given in Fig. 3, and those for triangular channels are given in Fig. 6. For channels with sloping side walls Eq. 22 is suggested. With the actual value of G exceeding the value determined from Eq. 21, there will be supercritical flow in part of the channel. Under these conditions, the solution can be obtained from Eqs. 24*b*, 25, and 26. The surface profile in the supercritical sections is shown in Fig. 7.

Experimental verification of the theory has been successfully carried out with model channels. Several examples are presented to illustrate the methods of design.

APPENDIX. NOTATION

The following symbols, adopted for use in this paper and for the guidance of discussers, conform essentially with American Standard Letter Symbols for Hydraulics (ASA Z10.2—1942), prepared by a Committee of the American Standards Association, with Society representation and approved by the Association in 1942:

A = cross-sectional area;
A_t = total wall and bottom area;
a_1 = area (Fig. 2);
a_2 = imaginary area (Fig. 5);
$B = \sqrt{a_2/A_o}$;
b = width of the water surface;
C = Chezy coefficient;
c = subscript used to denote the section with critical flow;
D = distance used to locate the centroid of areas a_1 and a_2;
$E = \sqrt{a_2/A_c}$;
F = Froude number $= \sqrt{\frac{Q^2\, b}{g\, A^3}}$;
$G = \dfrac{S_0\, L}{\frac{A_o}{b_o}}$;
h = height (Figs. 2 and 5);
$h_f = \dfrac{Q^2_o\, P_o\, L}{C^2_o\, A^3_o}$;
$I = \dfrac{4 + \alpha}{8 + 3\,\alpha}$;
$J = \dfrac{2 + \alpha}{6 + 5\,\alpha}$;
k = ratio of the width to the height of the channel shown in Fig. 5;
L = length of the channel;

o = subscript denoting the outlet end of the channel;
P = wetted perimeter;
Q = rate of discharge;
q = the additional discharge per unity length;
r = subscript denoting a rectangular section;
S_0 = slope of the channel bed;
t = subscript denoting a triangular section;
u = subscript denoting the upstream end of the channel;
v' = component of the velocity of the added water in the direction of the channel flow;
x = distance from the upstream end of the channel;
y = depth of flow (Figs. 2 and 5);
$\bar{y}$ = vertical distance from the centroid of water cross section to the water surface;
$\alpha = \dfrac{h_f}{\mathsf{F}^2_o\, y_o}$; and
ρ = mass per unit volume of water.

PROCEEDINGS-SEPARATES

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[a]Discussions grouped by Divisions.

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